

# ENTIRE DOWNWARD TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN MINKOWSKI SPACE

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**ABSTRACT.** In this paper, we study entire translating solutions  $u(x)$  to a mean curvature flow equation in Minkowski space. We show that if  $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$  is a strictly spacelike hypersurface, then  $\Sigma$  reduces to a strictly convex rank  $k$  soliton in  $\mathbb{R}^{k,1}$  (after splitting off trivial factors) whose “blowdown” converges to a multiple  $\lambda \in (0, 1)$  of a positively homogeneous degree one convex function in  $\mathbb{R}^k$ . We also show that there is nonuniqueness as the rotationally symmetric solution may be perturbed to a solution by an arbitrary smooth order one perturbation.

## 1. INTRODUCTION

Let  $\mathbb{R}^{n,1}$  be the Minkowski space with Lorentz metric

$$\bar{g} = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2.$$

We will say that a hypersurface  $\Sigma = \{(x, u(x)) | x \in \Omega\} \subset \mathbb{R}^{n,1}$  is strictly spacelike if  $u \in C^1(\Omega)$  and  $|Du| \leq c_0 < 1$  in  $\Omega$ .

Ecker and Huisken [5] studied the mean curvature flow with forcing term in Minkowski space and proved longtime existence. More specifically, they studied the equation

$$(1.1) \quad \frac{\partial u}{\partial t} = \sqrt{1 - |Du|^2} \left[ \operatorname{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) - \mathcal{H} \right],$$

where  $u = u(x, t)$  is the height function and  $\mathcal{H} = H(x)$  is the forcing term. Later, M. Aarons [1] proved the following convergence result.

**Theorem 1.1.** ([1]) *Let  $M_0$  be a smooth spacelike hypersurface with bounded curvature. Suppose  $M_0$  never intersects future null infinity  $I^+$  or past null infinity  $I^-$ . Then  $M_t$  converges under the flow (1.1) to a convex downward translating soliton, that is, an entire solution of  $H = c + a \frac{1}{\sqrt{1 + |\nabla u|^2}}$ ,  $a < 0$ .*

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After a rescaling, we may assume  $a = -1$  and consider  $H = C - \frac{1}{\sqrt{1-|Du|^2}}$ , where  $C > 1$  is a constant. We obtain

$$(1.2) \quad \operatorname{div} \left( \frac{\nabla u}{\sqrt{1-|Du|^2}} \right) = C - \frac{1}{\sqrt{1-|Du|^2}}$$

or in nondivergence form

$$(1.3) \quad \left( \delta_{ij} + \frac{u_i u_j}{1-|Du|^2} \right) u_{ij} = C \sqrt{1-|Du|^2} - 1.$$

Notice that any  $u = \vec{v} \cdot x + b$ ,  $|\vec{v}| = \sqrt{1 - \frac{1}{C^2}}$  (a maximal hypersurface) is a solution of (1.3). The existence of a unique (up to translation) radial symmetric solution of (1.3) was shown by Ju, Lu and Jian [9].

Aarons [1] in fact conjectured that any solution  $u$  of (1.2) is either rotationally symmetric about some point  $x_0$  or is a hyperplane. However, this conjecture is not correct. Let  $x' = (x_1, \dots, x_k)$  and set  $u(x) = \sum_{i=k+1}^n a_i x_i + h(x')$  where  $h$  is strictly convex in  $x'$ . Then  $u$  satisfies (1.3) if and only if  $h$  satisfies

$$(1.4) \quad \sum_{i,j=1}^k \left( \delta_{ij} + \frac{h_i h_j}{1-|a|^2-|Dh|^2} \right) h_{ij} = C \sqrt{1-|a|^2} \sqrt{1 - \frac{|Dh|^2}{1-|a|^2}} - 1.$$

Now let  $\tilde{h} = \frac{1}{\lambda^2} h(\lambda x)$ ,  $\tilde{C} = \lambda C > 1$  with  $\lambda = \sqrt{1-|a|^2}$ . Then  $\tilde{h}$  satisfies

$$(1.5) \quad \sum_{i,j=1}^k \left( \delta_{ij} + \frac{\tilde{h}_i \tilde{h}_j}{1-|D\tilde{h}|^2} \right) \tilde{h}_{ij} = \tilde{C} \sqrt{1-|D\tilde{h}|^2} - 1.$$

Thus  $\tilde{h}$  is a rank  $k$  solution of (1.3) in  $\mathbb{R}^k$  with  $C$  replaced by  $\tilde{C} = \lambda C > 1$ .

In fact, we will show a splitting theorem analogous to what Choi-Treibergs [4] proved for spacelike constant mean curvature hypersurfaces.

**Theorem 1.2.** *Let  $u$  be a strictly spacelike solution of (1.2) and let  $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$  be the graph of  $u$ . Then  $\Sigma$  is convex with uniformly bounded second fundamental form. Moreover after an  $R^{n,1}$  rigid motion,  $R^{n,1}$  splits as a product  $R^{k,1} \times R^{n-k}$  such that  $\Sigma$  also splits as a product  $\Sigma^k \times R^{n-k}$  where  $\Sigma^k = R^{n,1} \cup R^{k,1}$  is a strictly convex graphical solution in  $\mathbb{R}^{k,1}$*

Thus, it is natural to ask if Aarons conjecture correct for  $u$ , a strictly convex solution of (1.2) in  $\mathbb{R}^n$ ? In other words, is  $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$  rotationally symmetric? The answer is no.

**Theorem 1.3.** *Let  $f \in C^2(S_{\tilde{C}}^{n-1})$ ,  $\tilde{C} = \sqrt{1 - (\frac{1}{C})^2}$ . Then there exists an entire strictly spacelike hypersurface  $u$  satisfying equation (1.2) such that*

$$u(x) \rightarrow \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x) \text{ as } |x| \rightarrow \infty.$$

As in the work of Treibergs [12] and Choi-Treibergs [4], the blow-down of a convex strictly spacelike solution  $V_u = \lim_{r \rightarrow \infty} \frac{u(rx)}{r}$  converges uniformly on compact subsets to the space  $\tilde{C}\mathcal{Q}$  of convex homogeneous degree one convex functions whose gradient has magnitude  $\tilde{C}$  wherever defined. It was shown in [4] that the space  $\mathcal{Q}$  is in one to one correspondence with the set of *lightlike* directions

$$L_u := \{x \in S^{n-1} : V_u(x) = 1\}.$$

It may be possible that any cone in  $\tilde{C}\mathcal{Q}$  arises as the blow-down of a solution to (1.2) but we have not shown this.

An outline of the paper is as follows. In section 2 we show the strictly space like assumption implies that the graph  $\Sigma$  is mean convex. Then in section 3 we show  $\Sigma$  is in fact convex and then prove the splitting Theorem 1.2. In section 4 we study the blow-down  $V_u$  and finally in section 5 following [12], we construct counterexamples for the radial cone in  $\tilde{C}\mathcal{Q}$  and prove Theorem 1.3

## 2. STRICTLY SPACELIKE IMPLIES MEAN CONVEX

Let  $a^{ij} = \delta_{ij} + \frac{u_i u_j}{w^2}$ , where  $w = (1 - |Du|^2)^{1/2}$ ; then equation (1.3) becomes

$$(2.1) \quad a^{ij} u_{ij} = Cw - 1$$

Then  $w_i = -\frac{u_k u_{ki}}{w}$  and

$$(2.2) \quad \begin{aligned} w_{ij} &= -\frac{u_{ki} u_{kj}}{w} - \frac{u_k u_{kij}}{w} + \frac{u_k u_{ki} w_j}{w^2} \\ &= -\frac{u_k u_{kij}}{w} - \frac{1}{w} \left( u_{ki} u_{kj} + \frac{u_k u_{ki} u_l u_{lj}}{w^2} \right) \\ &= -\frac{u_k u_{kij}}{w} - \frac{1}{w} a^{kl} u_{ki} u_{lj}. \end{aligned}$$

**Lemma 2.1.** *Suppose  $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$  is a strictly spacelike hypersurface, and  $u(x)$  satisfies equation (1.3). Then  $\Sigma$  is mean convex, that is  $H \geq 0$ .*

*Proof.* We differentiate equation (2.1) with respect to  $x_k$  to obtain

$$(2.3) \quad (a^{ij})_k u_{ij} + a^{ij} u_{ijk} = Cw_k.$$

Since

$$\begin{aligned}
 (a^{ij})_k u_{ij} &= \left( \frac{u_{ik} u_j}{w^2} + \frac{u_i u_{jk}}{w^2} - 2 \frac{u_i u_j w_k}{w^3} \right) u_{ij} \\
 &= \frac{2}{w} \left( \frac{u_{ik} u_j}{w} + \frac{u_i u_j u_l u_{lk}}{w^3} \right) u_{ij} \\
 (2.4) \quad &= \frac{2}{w} \left( -w_i u_{ik} - \frac{u_i u_l u_{lk}}{w^2} w_i \right) \\
 &= -\frac{2}{w} \left( \delta_{ij} + \frac{u_i u_j}{w^2} \right) w_i u_{kj} \\
 &= -\frac{2}{w} a^{ij} w_i u_{kj},
 \end{aligned}$$

and  $u_{ijk} = u_{kij}$ , this gives

$$(2.5) \quad a^{ij} u_{kij} - \frac{2}{w} a^{ij} w_i u_{kj} = C w_k.$$

Multiplying (2.5) by  $\frac{u_k}{w}$  and using  $\frac{u_k u_{kij}}{w} = -w_{ij} - \frac{1}{w} a^{kl} u_{ki} u_{lj}$ , we obtain

$$(2.6) \quad a^{ij} w_{ij} - 2 \frac{a^{ij} w_i w_j}{w} + C \frac{u_k}{w} w_k = -\frac{1}{w} a^{ij} a^{kl} u_{ki} u_{lj}.$$

We now observe that since  $|A|^2 = \frac{1}{w^2} a^{ij} a^{kl} u_{ki} u_{lj}$  we can rewrite (2.6) as

$$(2.7) \quad a^{ij} \left( \frac{1}{w} \right)_{ij} + C \frac{u_k}{w} \left( \frac{1}{w} \right)_k = |A|^2 \frac{1}{w}.$$

The Omori-Yau maximum principle (see for example [13], [11]) implies that  $\frac{1}{w}$  achieves its maximum at infinity and moreover, there exists a sequence  $\{P_N\}$  such that  $\frac{1}{w}(P_N) \rightarrow \sup \frac{1}{w}$ ,  $|\nabla(\frac{1}{w})|(P_N) < 1/N$ , and  $(\frac{1}{w})_{ij}(P_N) \geq -1/N \delta_{ij}$ . Therefore,

$$(2.8) \quad \frac{1}{n} H^2 \frac{1}{w} \leq |A|^2 \frac{1}{w} \leq \frac{C_1}{N} \frac{1}{w} \quad \text{at } P_N.$$

Thus  $H(P_N) \rightarrow 0$  at infinity. Since  $H = C - \frac{1}{w}$  we obtain  $\inf H = 0$ .  $\square$

### 3. MEAN CONVEXITY IMPLIES CONVEXITY AND CONSTANT RANK

In this section, we will use ideas due to Hamilton [7] to prove that under the strictly spacelike assumption,  $\Sigma$  is in fact convex. We use the following approximation of Heidusch[8].

**Definition 3.1.** The  $\delta$ -approximation to the function  $\min(x_1, x_2)$  is given by

$$\mu_2(x_1, x_2) = \frac{x_1 + x_2}{2} - \sqrt{\left( \frac{x_1 - x_2}{2} \right)^2 + \delta^2}$$

for any  $\delta > 0$ . The  $\delta$ -approximation to the function  $\min(x_1, x_2, \dots, x_n)$  is defined recursively by

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \mu_2(x_i, \mu_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n))$$

The following lemma is elementary (see [1]).

**Lemma 3.2.** *For every  $\delta > 0$  and  $n \geq 2$  we have:*

1.  $\mu_n$  is smooth, symmetric, monotonically increasing and concave.
2.  $\frac{\partial \mu_n}{\partial x_i} \leq 1$ .
3.  $\min(x_1, \dots, x_n) - n\delta \leq \mu_n \leq \min(x_1, \dots, x_n)$ .
4. For  $x \in \mathbb{R}^n$  we have

$$\mu_n \leq \sum_{i=1}^n \frac{\partial \mu_n}{\partial x_i} x_i \leq \mu_n + n\delta,$$

$$\text{and } \sum_{i=1}^n \frac{\partial \mu_n}{\partial x_i} x_i^2 \geq \mu_n^2 - n\delta^2 - \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |x_i + x_j|.$$

**Lemma 3.3.** *Assume  $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$  is mean convex, and  $u$  satisfies equation (1.3). Then the principal curvatures of  $A$  are nonnegative, i.e.  $\Sigma$  is convex.*

*Proof.* Let  $p$  be a fixed point in  $\Sigma$  (we may assume  $p = (0, 0)$ ) and let  $r$  be the distance function from  $p$  restricted to the geodesic ball  $B^\Sigma(p, a)$  of radius  $a$  centered at  $p$  (in the induced metric on  $\Sigma$ ). Let  $f(x) = |A|^2 = \sum_{i,j} h_{ij}^2$ . By a well-known calculation (see equation (2.24) of [3])

$$(3.1) \quad \frac{1}{2} \Delta \left( \sum_{i,j} h_{ij}^2 \right) = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} H_{ij} + \left( \sum_{i,j} h_{ij}^2 \right)^2 - \left( \sum_{i,j,k} h_{ij} h_{jm} h_{mi} \right) H$$

$$(3.2) \quad \begin{aligned} H_{ij} &= \nabla_i \nabla_j H = \nabla_i (\nabla_j \langle \nu, e_{n+1} \rangle) \\ &= \nabla_i \langle h_{jk} \tau_k, e_{n+1} \rangle = -h_{ijk} u_k - h_{ik} h_{jk} \nu^{n+1}. \end{aligned}$$

In the following, we will denote  $\nu^{n+1}$  by  $V$ .

$$(3.3) \quad \begin{aligned} \frac{1}{2} \Delta f &= \sum_{i,j,k} h_{ijk}^2 - h_{ij} h_{ijk} u_k - h_{ij} h_{ik} h_{jk} V + f^2 - \left( \sum_{i,j,m} h_{ij} h_{jm} h_{mi} \right) (C - V) \\ &\geq -\frac{f(V^2 - 1)}{4} + f^2 - C f^{3/2} \\ &\geq -\frac{C^2}{4} f + f^2 - C f^{3/2} \geq \frac{1}{2} f^2 - C_1 f \end{aligned}$$

Let  $\eta(x) = a^2 - r^2$  and set  $g = \eta^2 f$ . Then in  $B^\Sigma(p, a)$ ,

$$(3.4) \quad \frac{1}{2}(\eta^{-2}g)^2 \leq C_1\eta^{-2}g + \Delta(\eta^{-2}g) = C_1\eta^{-2}g + \eta^{-2}\Delta g - 2\eta^{-3} \langle \nabla\eta, \nabla g \rangle + g\Delta(\eta^{-2}).$$

At the point  $\bar{x}$  where  $g$  assumes its maximum,  $\nabla g = 0$  and  $\Delta g \leq 0$ . Since  $R_{ii} \geq -\frac{H^2}{4} \geq -\frac{C^2}{4}$ , we have by Lemma 1 of [13] that  $\Delta r^2 \leq C_3(1 + r^2)$ . Hence at  $\bar{x}$ ,

$$(3.5) \quad \begin{aligned} \frac{1}{2}g^2 &\leq C_1\eta^2g + g\eta^4\Delta(\eta^{-2}) = C_1\eta^2g - 2g\eta\Delta\eta + 6g|\nabla\eta|^2 \\ &\leq C_2g(a^4 + 2a^2|\Delta r^2| + 24r^2) \end{aligned}$$

It follows that  $g(\bar{x}) \leq C_5a^4$ . Therefore, let  $a \rightarrow \infty$  we get,  $|A|^2 \leq C_6$ .

Next we will show that the smallest principal curvature  $\lambda_{\min}$  of  $\Sigma$  is nonnegative. Let  $\mu_n(\lambda_1, \dots, \lambda_n) = F(\gamma^{ik}h_{kl}\gamma^{lj})$ , assume  $\mu_n$  achieves its minimum at an interior point  $x_0$ . Then at this point we have

$$(3.6) \quad \begin{aligned} \Delta\mu_n &= F^{ij}h_{ijkk} + F^{rl, st}h_{rlk}h_{stk} \\ &= F^{ij}(H_{ij} - Hh_{ij}^2 + h_{ij}h_{kk}^2) + F^{rl, st}h_{rlk}h_{stk} \\ &\leq F^{ij}\nabla_k h_i^j \langle \tau_k, e_{n+1} \rangle - F^{ij}h_i^k h_{jk}\nu^{n+1} - HF^{ij}h_{ij}^2 + (\mu_n + n\delta)|A|^2 \\ &\leq \langle \nabla_k \mu_n, e_{n+1} \rangle - \mu_n^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |\lambda_i + \lambda_j| \\ &\quad + H \left( -\mu_n^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |\lambda_i + \lambda_j| \right) + (\mu_n + n\delta)|A|^2. \end{aligned}$$

Thus,

$$(3.7) \quad 0 \leq -\mu_n^2 + (\mu_n + n\delta)|A|^2 + n\delta^2 + \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |\lambda_i + \lambda_j|.$$

Letting  $\delta \rightarrow 0$  we find,

$$(3.8) \quad \lambda_{\min}^2 \leq \lambda_{\min}|A|^2,$$

which implies that  $\lambda_{\min} \geq 0$ .

Since we have already proven that  $|A|^2$  is bounded, we can again apply the Omori-Yau maximum principle (this time on  $\Sigma$ ) and show that, if  $\mu_n$  achieves its minimum at infinity then  $\mu_n \geq 0$ . This completes the proof that mean convexity implies convexity.  $\square$

Now that we have proved convexity, we prove the splitting Theorem 1.2 of the introduction.

**Proof of Theorem 1.2.** Suppose that for some unit vector  $\vec{v}$  and some  $x_0 \in \mathbb{R}^n$ ,  $D_{\vec{v}}^2 u(x_0) = 0$ . Applying an isometry (boost transformation) of  $\mathbb{R}^{n,1}$ , may assume  $x_0 = 0$ ,  $\vec{v} = e_n$ ,  $Du(0) = 0$ ,  $u_{nn}(0) = 0$  and  $u_{ij}$  is nonnegative. Rewrite (1.3) as

$$(3.9) \quad \Delta u = -\frac{u_i u_j}{1 - |Du|^2} u_{ij} + C \sqrt{1 - |Du|^2} - 1$$

Differentiating (3.9) twice in the  $x_n$  direction, we can apply the argument of Korevaar (a special case of [10]) exactly as in Theorem 3.1 of Choi-Treibergs [4] to conclude  $u_{nn} \equiv 0$  and  $\Sigma$  is ruled by lines parallel to the  $x_n$  axis. Therefore  $\Sigma = \Sigma^{n-1} \times \mathbb{R}^1$  and also  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$ . Therefore  $u = ax_n + h(x')$ ,  $x' = (x_1, \dots, x_{n-1})$  where

$$(3.10) \quad \sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{h_i h_j}{1 - |a|^2 - |Dh|^2}) h_{ij} = C \sqrt{1 - |a|^2} \sqrt{1 - \frac{|Dh|^2}{1 - |a|^2}} - 1.$$

Now let  $\tilde{h} = \frac{1}{\lambda^2} h(\lambda x)$ ,  $\tilde{C} = \lambda C > 1$  with  $\lambda = \sqrt{1 - |a|^2}$ . Then  $\tilde{h}$  satisfies

$$(3.11) \quad \sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{\tilde{h}_i \tilde{h}_j}{1 - |D\tilde{h}|^2}) \tilde{h}_{ij} = \tilde{C} \sqrt{1 - |D\tilde{h}|^2} - 1.$$

Proceeding inductively completes the proof of Theorem 1.2.

#### 4. THE ASYMPTOTIC CONE AT INFINITY.

In this section, we will study the asymptotic behavior of  $u$  at infinity.

**Proposition 4.1.** *Let  $u$  be a convex space like solution of (1.3). Assume  $u(0) = 0$  and denote  $u^h(x) = \frac{u(hx)}{h}$ . Define  $V_u(x) = \lim_{h \rightarrow \infty} u^h(x)$  then  $V_u(x)$  exists for all  $x$  and is a positively homogeneous degree one convex function. Moreover for all  $x \in \mathbb{R}^n$  and  $\delta > 0$  there exists  $y \in \mathbb{R}^n$  so that  $|y - x| = \delta$  and  $|V_u(x) - V_u(y)| = \sqrt{1 - \frac{1}{C^2}} \delta$ . In particular  $|DV_u(x)| = \sqrt{1 - \frac{1}{C^2}}$  at every point of differentiability of  $V_u(x)$ .*

*Proof.* Note that since  $u$  is convex,  $0 = u(0) \geq u(hx) - \sum_{i=1}^n h x_i u_{x_i}(hx)$  we get  $\frac{d}{dh} u^h(x) \geq 0$ . Then  $V_u(x)$ , the projective boundary values (blow-down) of  $u$  at infinity in the terminology of Treibergs [12] and Choi-Treibergs [4], is well-defined, strictly spacelike, convex on  $\mathbb{R}^n$  and satisfies

$$V_u(\lambda x) = \lambda V_u(x), \quad \lambda > 0,$$

$$|V_u(x) - V_u(y)| \leq \sqrt{1 - \frac{1}{C^2}} |x - y|.$$

Claim: for all  $x \in \mathbb{R}^n$  and  $\delta > 0$  there exists  $y \in \mathbb{R}^n$  so that  $|y - x| = \delta$  and  $|V_u(x) - V_u(y)| = \sqrt{1 - \frac{1}{C^2}} \delta$ . Suppose the claim is false. Then there exists  $x \in \mathbb{R}^n$  and  $\varepsilon > 0$  such that

$$V_u(y) \leq V_u(x) + (1 - 2\varepsilon) \sqrt{1 - \frac{1}{C^2}} \delta \quad \forall y \in \partial B(x, \delta) .$$

Since  $u^h(x) \rightarrow V_u(x)$  uniformly on compact subsets, we may choose  $h_0$  large so that for all  $h > h_0$ ,

$$u^h(y) \leq V_u(x) + (1 - \varepsilon) \sqrt{1 - \frac{1}{C^2}} \delta \quad \forall y \in \partial B(x, \delta) .$$

Now  $u^h(y)$  satisfies

$$(4.1) \quad H^h := \operatorname{div} \left( \frac{Du^h}{\sqrt{1 - |Du^h|^2}} \right) = h \left( C - \frac{1}{\sqrt{1 - |Du^h|^2}} \right) \geq 0 \quad \text{in } B(x, \delta)$$

We now make use of the radial solutions of the maximal surface equation  $H = 0$  introduced by Bartnik and Simon [2]. Consider the barrier

$$w(y) = V_u(x) + (1 - \varepsilon) \sqrt{1 - \frac{1}{C^2}} \delta + \int_0^{|y-x|} \frac{h}{\sqrt{t^{2n-2} + h^2}} dt - \sqrt{1 - \frac{1}{C^2}} \int_0^\delta \frac{h}{\sqrt{t^{2n-2} + h^2}} dt .$$

Note that on  $\partial B(x, \delta)$ ,

$$w(y) - u^h(y) \geq (1 - \sqrt{1 - \frac{1}{C^2}}) \int_0^\delta \frac{h}{\sqrt{t^{2n-2} + h^2}} dt > 0 .$$

Hence by the maximum principle,  $u^h(y) < w(y)$  in  $B(x, \delta)$ . In particular at  $y = x$ ,

$$u^h(x) < V_u(x) + (1 - \varepsilon) \sqrt{1 - \frac{1}{C^2}} \delta - \sqrt{1 - \frac{1}{C^2}} \int_0^\delta \frac{h}{\sqrt{t^{2n-2} + h^2}} dt .$$

Now let  $h \rightarrow \infty$  to conclude  $V_u(x) \leq V_u(x) - \varepsilon \sqrt{1 - \frac{1}{C^2}} \delta$ , a contradiction, so the claim is proven and the proposition is complete.  $\square$

## 5. CONSTRUCTION OF COUNTEREXAMPLES.

We will follow Treiberg's idea (see [12]) to construct counterexamples, more precisely we will construct solution to equation (1.2) such that  $u(x) \rightarrow \tilde{C}|x| - \frac{n-1}{C^2} \log|x| + f\left(\frac{\tilde{C}x}{|x|}\right)$ , as  $|x| \rightarrow \infty$ , where  $\tilde{C} = \sqrt{1 - \frac{1}{C^2}}$  and  $f \in C^2(S_{\tilde{C}}^{n-1})$ .



We extend the function  $f$  to  $\mathbb{R}^n \setminus \{0\}$  by defining  $f(\tilde{C}x) = f\left(\frac{\tilde{C}x}{|x|}\right)$ . Since  $f \in C^2$ , we have for all  $x, y \in S^{n-1}$ :

$$(5.1) \quad |f(\tilde{C}x) - f(\tilde{C}y) - Df(\tilde{C}y)(\tilde{C}x - \tilde{C}y)| \leq M|\tilde{C}x - \tilde{C}y|^2 = -2\tilde{C}My \cdot (\tilde{C}x - \tilde{C}y).$$

Let  $p_1(\tilde{C}y) = Df(\tilde{C}y) + 2M\tilde{C}y$  and  $p_2(\tilde{C}y) = Df(\tilde{C}y) - 2M\tilde{C}y$ , so that

$$(5.2) \quad p_1(\tilde{C}y) \cdot (\tilde{C}x - \tilde{C}y) \leq f(\tilde{C}x) - f(\tilde{C}y) \leq p_2(\tilde{C}y) \cdot (\tilde{C}x - \tilde{C}y).$$

Now let  $\psi(x)$  denote the rotationally symmetric solution to (1.2) (see [9]). We know that  $\psi(x) \rightarrow \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + o(1)$  as  $|x| \rightarrow \infty$ . Let  $z_1(x; \tilde{C}y) = f(\tilde{C}y) - p_1(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_1(\tilde{C}y))$  and  $z_2(x; \tilde{C}y) = f(\tilde{C}y) - p_2(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_2(\tilde{C}y))$ . Then by equation (5.2) we have

$$(5.3) \quad f(\tilde{C}x) \geq z_1(rx; \tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2} \log r \text{ as } r \rightarrow \infty, x, y \in S^{n-1},$$

and

$$(5.4) \quad f(\tilde{C}x) \leq z_2(rx; \tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2} \log r \text{ as } r \rightarrow \infty, x, y \in S^{n-1}.$$

Therefore,

$$(5.5) \quad \begin{aligned} \lim_{r \rightarrow \infty} z_1(rx; \tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2} \log r &\leq f(\tilde{C}x) \\ &\leq \lim_{r \rightarrow \infty} z_2(rx; \tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2} \log r \end{aligned}$$

for  $x \in S^{n-1}$ .

Let  $q_1(x) = \sup_{y \in S^{n-1}} z_1(x; \tilde{C}y)$  and  $q_2(x) = \inf_{y \in S^{n-1}} z_2(x; \tilde{C}y)$ . Then,  $q_1(x) \leq q_2(x)$  and  $q_i(x)$  ( $i=1,2$ ) tends to  $f(\tilde{C}x) + \tilde{C}r - \frac{n-1}{C^2} \log r$  as  $r \rightarrow \infty$ .

**Lemma 5.1.** *There exists a smooth solution  $u$  to the Dirichlet problem*

$$(5.6) \quad \begin{cases} a^{ij}u_{ij} - Cw + 1 = 0 & \text{in } G \\ u = 0 & \text{on } \partial G \end{cases}$$

where  $G$  is a convex  $C^{2,\alpha}$  domain in  $\mathbb{R}^n$ .

*Proof.* Let  $d = \text{diam}(G)$  be the diameter of  $G$ . For any  $y \in \partial G$ , we can choose coordinates such that  $y = (y_1, 0, \dots, 0)$  and  $G \subset \{x \mid |x_1| \leq y_1\}$ , where  $0 < y_1 \leq d/2$ . Let  $\underline{u}(x) = \tilde{C}x_1 - \tilde{C}y_1$ ,  $\bar{u} \equiv 0$ . Then  $\underline{u} \leq \bar{u}$  in  $G$  and  $\underline{u}$  satisfies

$$(5.7) \quad a^{ij}\underline{u}_{ij} - Cw_{\underline{u}} + 1 = 0$$

By the maximum principle, any solution  $u$  to the Dirichlet problem (5.6) satisfies

$$(5.8) \quad \underline{u} \leq u \leq \bar{u} \text{ on } G.$$

so  $|Du(x)| \leq \tilde{C}$  on  $\partial\bar{G}$ . Combined with equation (2.7) we conclude that

$$(5.9) \quad |Du(x)| \leq \tilde{C} \text{ on } \bar{G}.$$

Now it is standard (see [6]) to prove that a smooth solution  $u \in C^{2,\alpha}(\bar{G})$  exists.  $\square$

Finally, we will find a sandwiched solution  $u$  such that

$$q_1 \leq u < q_2.$$

Let  $\phi$  be a strictly spacelike hypersurface  $q_1 \leq \phi < q_2$  so that  $\phi(0) = q_1(0)$  and  $G_m = \phi^{-1}((-\infty, m))$  is a convex domain with  $C^{2,\alpha}$  boundary. By lemma 5.1 we know there is an analytic solution  $u_m$  to the Dirichlet problem

$$(5.10) \quad \begin{aligned} a^{ij}u_{ij} - Cw + 1 &= 0 \text{ on } G_m \\ u &= m \text{ on } \partial G_m. \end{aligned}$$

Therefore, we find a sequence of finite solutions  $u_m$  with  $q_1 \leq u_m < q_2$  defined on convex domains  $G_m$  which exhaust  $\mathbb{R}^n$ .

Next, let  $K$  be a compact subset of  $\mathbb{R}^n$ . Then, by equation (5.9) there are constants  $r_1 < r_2$  so that for sufficiently large  $m$  we have

$$\text{dist}_m(0, x) < r_1, \text{ for all } x \in K,$$

$$\text{dist}_m(0, x) < r_2, \text{ for all } x \in \partial G_m,$$

where  $\text{dist}_m(0, x)$  is the intrinsic distance between the points  $(0, u_m(0))$  and  $(x, u_m(x))$  on  $\Sigma_m = \{(x, u_m(x)) | x \in G_m\}$ .

At last, following the proof of Lemma 3.3, we find  $u_m$  has uniform  $C^3$  bounds on compact subsets. Hence, a subsequence can be extracted that converges to a global solution of equation (1.2). Moreover,  $\lim_{m_j \rightarrow \infty} u_{m_j} = u$  satisfies  $u(x) \rightarrow \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x)$ . Thus we have proved

**Theorem 5.2.** *Let  $f \in C^2(S_{\tilde{C}}^{n-1})$ . Then there exists an entire strictly spacelike hypersurface  $u$  satisfying equation (1.2) such that*

$$u(x) \rightarrow \tilde{C}|x| - \frac{n-1}{C^2} \log |x| + f(\tilde{C}x) \text{ as } |x| \rightarrow \infty.$$

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